Further Topics in Computational Learning Theory

Lecture Outline

- Brief Review of the PAC-Learning Model
- Sample Complexity for Infinite Hypothesis Spaces
  - "Shattering" Sets of Instances
  - Vapnik-Chervonenkis (VC) Dimension
  - Sample Complexity and VC Dimension
- The Mistake Bound Model of Learning
  - Mistake Bound for FIND-S
  - Mistake Bound for the Halving Algorithm
  - Optimal Mistake Bounds
  - Weighted-Majority Algorithm

Reading

Mitchell, Chapter 7.4-7.5

M. Anthony and N. Briggs *Computational Learning Theory*, Cambridge University Press, Cambridge, 1992.

Sample Complexity for Infinite Hypothesis Spaces

- Our initial central result on sample complexity for consistent learners over finite hypothesis spaces was:

$$m \geq \frac{1}{\varepsilon}(\ln |H| + \ln(1/\delta))$$

  I.e. m training examples are required to ensure with probability $1 - \delta$ that any consistent hypothesis will be correct with true error no more than ε.

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- Note that sample complexity here grows as the log of the size of the hypothesis space H.
- This result has two limitations:
  1. It leads to weak bounds (overestimates of sample complexity)
  2. It only applies to finite hypothesis spaces
- An alternative is to consider other measures for the complexity of H, aside from simply its size.
- One such measure is the **Vapnik-Chervonenkis (VC) dimension**. VC dimension:
  1. Allows tighter bounds on sample complexity
  2. Can be applied to infinite hypothesis spaces

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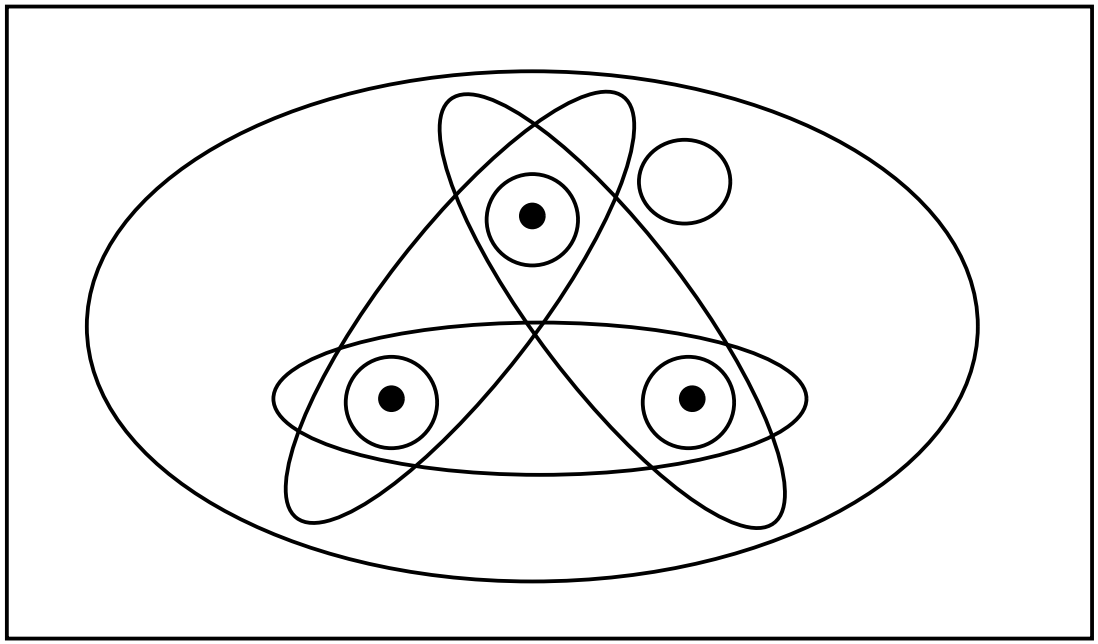
- Example: A set of 3 instances shattered by 8 hypotheses.

Instance space X



Vapnik-Chervonenkis (VC) Dimension

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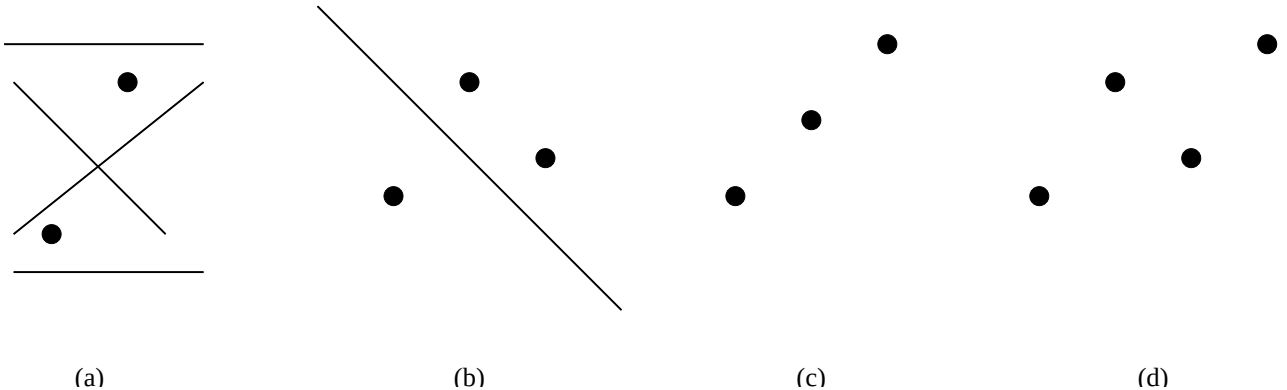
- The intuition behind VC dimension is that the larger the subset of X that can be shattered by H, the more expressive is H.

  Hence, VC dimension serves as an alternative measure of the complexity of H – alternative to simple size.

Vapnik-Chervonenkis (VC) Dimension: Example – Linear Decision Surfaces



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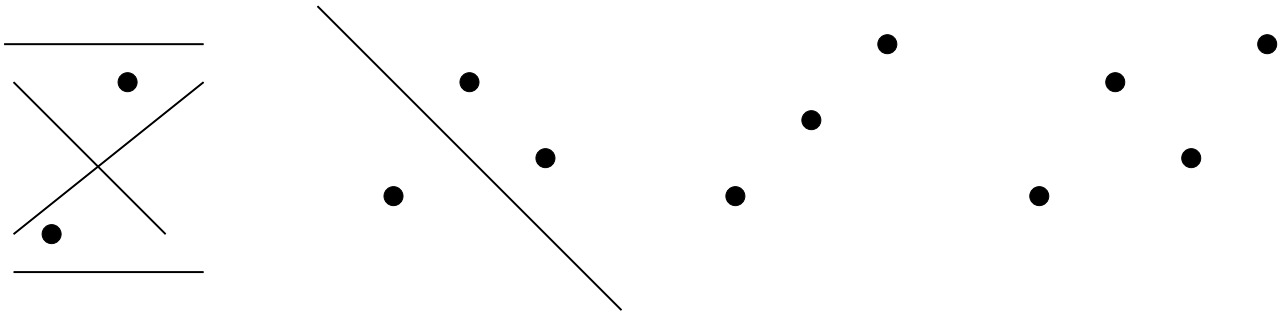

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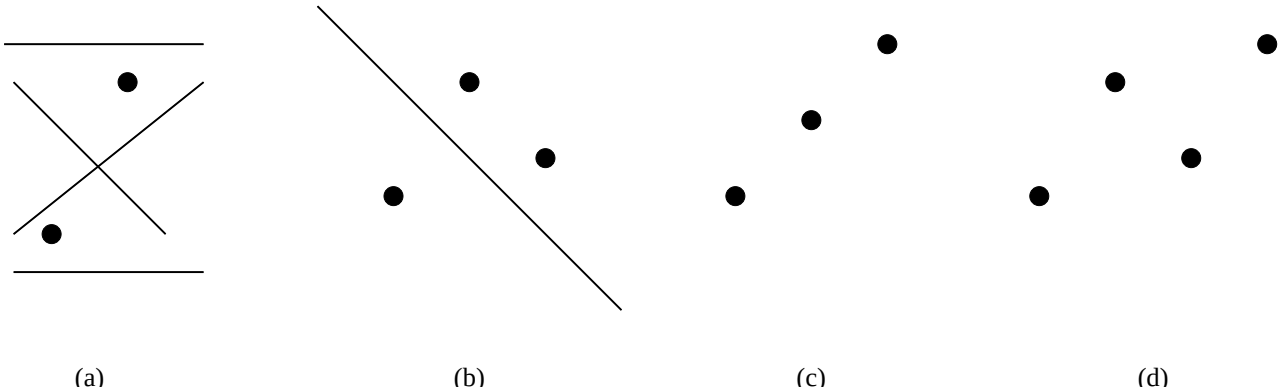

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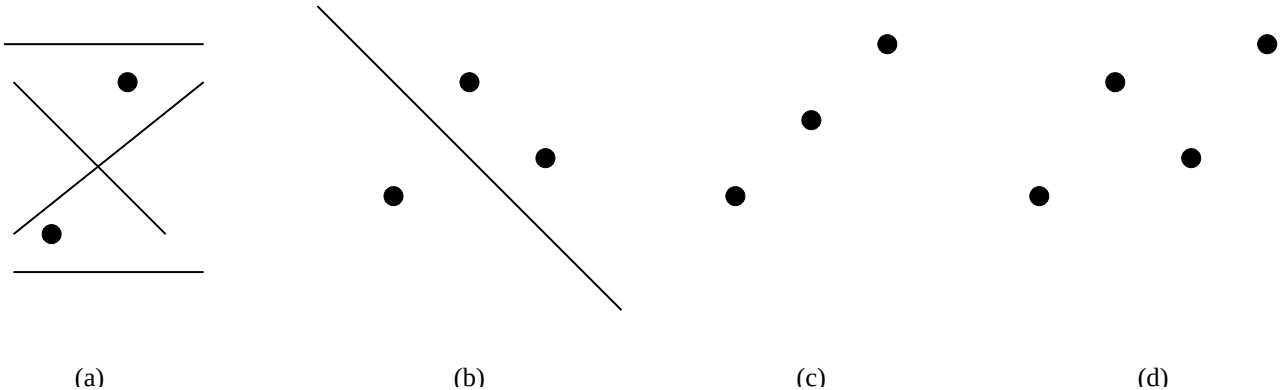

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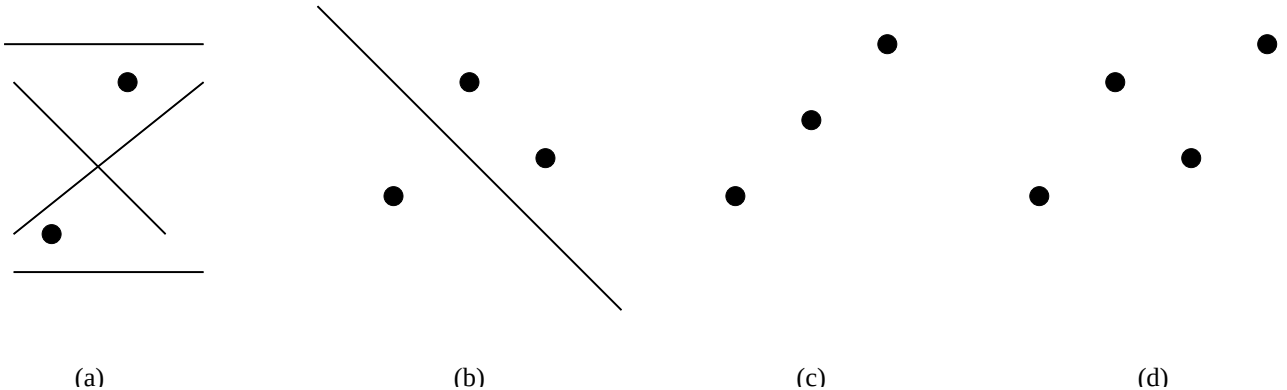

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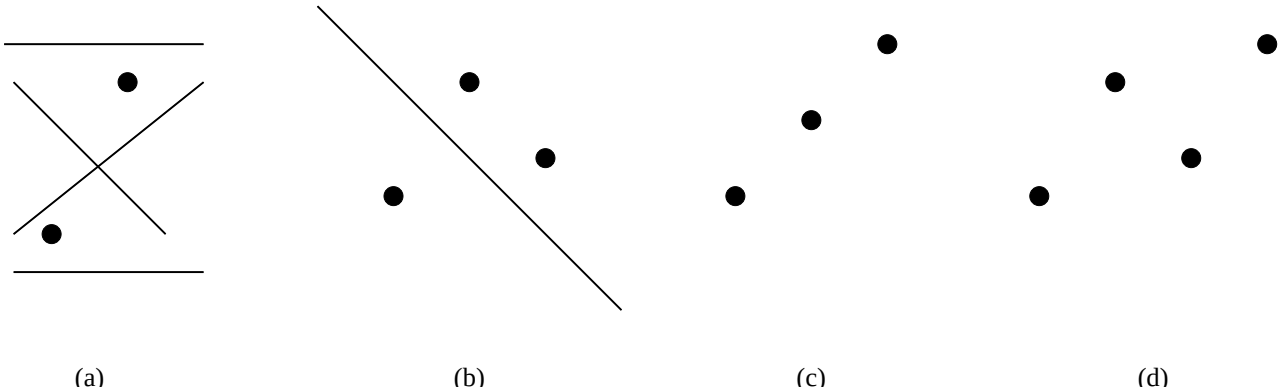

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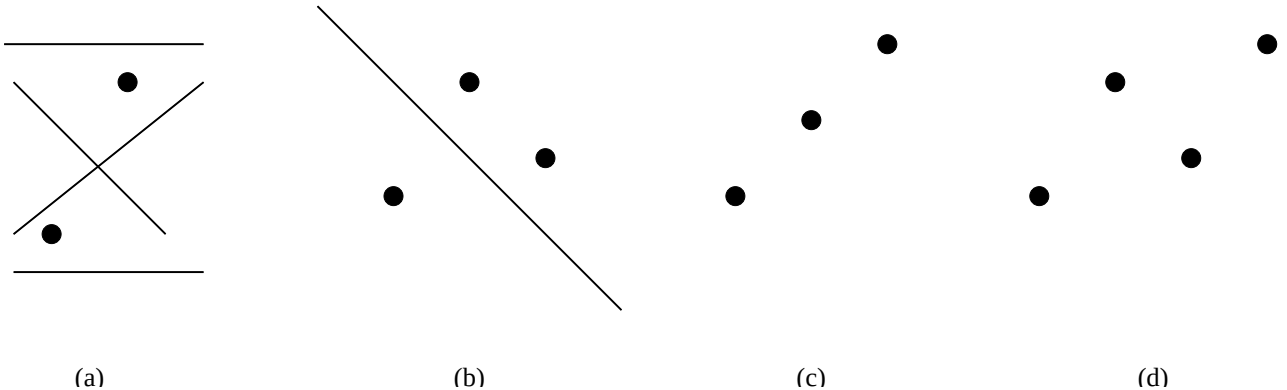

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- Therefore VC dimension of H ($VC(H)$) defined over X is 3.
- More generally, VC dimension of linear decision surfaces in an r dimensional space is $r+1$

Sample Complexity and VC Dimension

- Using VC dimension as a measure of hypothesis space complexity it is now possible to calculate a new **upper bound** on sample complexity.

  m randomly drawn examples suffice to ε-exhaust $VS_{H,D}$ with probability at least $(1-\delta)$, where

$$m \geq \frac{1}{\varepsilon}(4\log_2(2/\delta) + 8VC(H)\log_2(13/\varepsilon))$$

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- A **lower bound** on the number of training examples to ensure PAC-learnability of every concept in C has also been established.

  Theorem: Consider any concept class C such that $VC(C) \geq 2$, any learner L, and any $0 < \varepsilon < \frac{1}{8}$, and $0 < \delta < \frac{1}{100}$. Then there exists a distribution $\mathcal{D}$ and target concept C such that if L observes fewer examples than

$$max\left[\frac{1}{\varepsilon}log(1/\delta), \frac{VC(C)-1}{32\varepsilon}\right]$$

  then with probability at least δ, L outputs a hypothesis h having error $error_{\mathcal{D}} > \varepsilon$.

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- Mistake bound model can be of practical interest in settings where learning must take place during use of system, rather than in off-line training stage.

  Clearly want to minimise errors during this process.

Mistake Bound for FIND-S

- Consider FIND-S when H = conjunction of boolean literals (e.g. *Studies* $\wedge$ *Attends_Lectures* $\wedge \neg$ *Drinks*)

  FIND-S:

  – Initialize h to the most specific hypothesis $l_1 \wedge \neg l_1 \wedge l_2 \wedge \neg l_2 \ldots l_n \wedge \neg l_n$
  – For each positive training instance x
    * Remove from h any literal that is not satisfied by x
  – Output hypothesis h.

  How many mistakes before converging to correct h?

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  5. Each subsequent misclassified positive example will cause at least one further term to be removed.
  6. Therefore, in the worst case, the total number of mistakes will be $n+1$.

Mistake Bound for the Halving Algorithm

- Consider the Halving Algorithm:
  - Learn concept using version space CANDIDATE-ELIMINATION algorithm
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  5. In best case, majority is always right, minority hypotheses are removed from version space at each step, and HALVING algorithm converges to a single member version space making 0 errors.

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  I.e. $Opt(C)$ is the number of mistakes made

  - for the hardest concept in C
  - using the hardest training sequence
  - by the best algorithm

Optimal Mistake Bounds (cont)

- The following relationship has been proved to hold:

$$VC(C) \leq Opt(C) \leq M_{Halving}(C) \leq log_2(|C|).$$

Weighted-Majority Algorithm

- The WEIGHTED-MAJORITY algorithm is a generalisation of the HALVING algorithm.
  - Like the HALVING algorithm it takes a vote amongst alternative hypotheses (or even alternative learning algorithms).
  - Unlike the HALVING algorithm the votes are weighted, and instead of eliminating incorrect hypotheses, their weights are adjusted downwards after each mistake.

- The WEIGHTED-MAJORITY algorithm begins by assigning a weight of 1 to each predictor and then considers the training example sequence.

  Whenever a predictor misclassifies a training example its weight is decreased by multiplying it by a factor β (so, when $\beta = 0$, WEIGHTED-MAJORITY algorithm = HALVING algorithm).

Weighted-Majority Algorithm (cont)

- Two interesting features of WEIGHTED-MAJORITY algorithm
  1. Handles inconsistent/noisy data (hypotheses never eliminated, just down-weighted)
  2. Can establish mistake bounds in terms of the best of the predictors:

     Let D be any sequence of training examples, A any set of n prediction algorithms, k the minimum number of mistakes made by any algorithm in A for D.

     Then the number of mistakes made over D by the WEIGHTED-MAJORITY algorithm is at most

$$\frac{k log_2 \frac{1}{\beta} + log_2 n}{log_2 \frac{2}{1+\beta}}$$

     where $0 \leq \beta < 1$ is the weight adjustment factor applied after each error is encountered.

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- The *probably approximately correct (PAC)* learning model characterises concept classes, learners, and hypothesis spaces in terms of whether the learner can *probably* learn a hypothesis that is *approximately* correct.

- *Sample complexity*, the number of training examples required by a learner, can be quantitatively expressed as
  - a function of the size of the hypotheses space, for finite hypothesis spaces
  - a function of the size of the VC dimension of the hypotheses space (intuitively, the expressivity of the hypothesis space) for non-finite hypothesis spaces.

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  – General results here show that for an arbitrary concept class C, the best worst-case algorithm makes $Opt(C)$ mistakes where

  $$VC(C) \leq Opt(C) \leq log_2(|C|)$$

  – The WEIGHTED-MAJORITY algorithm, which combines the votes of multiple predictors, can be shown to have a mistake bound limited by that of the best of its predictors.